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Author: Janusz Morawiec

Citation style: Morawiec Janusz. (2015). Around a problem of Nicole Brillouët-Belluot, II. "Aequationes Mathematicae" (Vol. 89, no. 3 (2015), s. 625-627), doi 10.1007/s00010-013-0249-z



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Around a problem of Nicole Brillouët-Belluot, II

JANUSZ MORAWIEC

Abstract. For every $\alpha \in \mathbb{R}$ we determine all increasing bijections $f: (0, +\infty) \rightarrow (0, +\infty)$ such that $f(1) \neq 1$ and $f(x)f^{-1}(x) = x^\alpha$ for every $x \in (0, +\infty)$.

Mathematics Subject Classification (2010). Primary 39B22; Secondary 39B12, 26A18.

Keywords. Iterative functional equation, bijection, iterate, increasing solution.

In this note we answer Problem 3.8 posed in my article [4]. More precisely, given $\alpha \in \mathbb{R}$ we determine all increasing bijections $f: (0, +\infty) \rightarrow (0, +\infty)$ such that $f(1) \neq 1$ and

$$f(x)f^{-1}(x) = x^\alpha \quad \text{for every } x \in (0, +\infty). \quad (1.1)$$

Theorem 1. (i) If (1.1) admits a solution f which is an increasing bijection from $(0, +\infty)$ onto $(0, +\infty)$, then we have $\alpha > 2$. Let $\beta \in (1, +\infty)$ be the unique number satisfying $\beta + \frac{1}{\beta} = \alpha$.

If $f(1) = c \neq 1$ and $f_0 = f|_{[\min\{1, c\}, \max\{1, c\}]}$, then f_0 is continuous, $f_0(c) = c^\alpha$ and

$$\left(\frac{y}{x}\right)^{\frac{1}{\beta}} \leq \frac{f_0(y)}{f_0(x)} \leq \left(\frac{y}{x}\right)^\beta \quad (1.2)$$

for all $x, y \in [\min\{1, c\}, \max\{1, c\}]$ with $x < y$.

(ii) Conversely, let $c \in (0, 1) \cup (1, +\infty)$ and let $f_0: [\min\{1, c\}, \max\{1, c\}] \rightarrow \mathbb{R}$ be a continuous function such that $f_0(1) = c$, $f_0(c) = c^\alpha$ and (1.2) holds for all $x, y \in [\min\{1, c\}, \max\{1, c\}]$ with $x < y$. Then f_0 can be uniquely extended to an increasing bijection $f: (0, +\infty) \rightarrow (0, +\infty)$ satisfying (1.1).

Moreover:

- (a) if $c \in (0, 1)$, then $f(x) < \min\{x^\beta, x^{\frac{1}{\beta}}\}$ for every $x \in (0, +\infty)$;
- (b) if $c \in (1, +\infty)$, then $f(x) > \max\{x^\beta, x^{\frac{1}{\beta}}\}$ for every $x \in (0, +\infty)$.

Proof. (i) The case $f(1) = 1$ was treated in [4]. Then we have $\alpha > 0$ by Lemma 3.2 of [4].

Fix an increasing bijective solution $f: (0, +\infty) \rightarrow (0, +\infty)$ of (1.1) such that $f(1) \neq 1$.

Put $c = f(1)$ and $f_0 = f|_{[\min\{1, c\}, \max\{1, c\}]}$. Then f_0 is continuous, $f_0(1) = c$ and $f_0(c) = f^2(1) = [f(1)]^\alpha = c^\alpha$.

Fix $x, y \in [\min\{1, c\}, \max\{1, c\}]$ with $x < y$. Since all the iterates of f are increasing, by Lemma 2.3 of [4] we have

$$\frac{[f_0(x)]^{a_n}}{x^{a_{n-1}}} < \frac{[f_0(y)]^{a_n}}{y^{a_{n-1}}} \quad \text{and} \quad \frac{x^{a_{n+1}}}{[f_0(x)]^{a_n}} < \frac{y^{a_{n+1}}}{[f_0(y)]^{a_n}} \quad \text{for every } n \in \mathbb{N}.$$

Hence

$$\left(\frac{y}{x}\right)^{\frac{a_{n-1}}{a_n}} \leq \frac{f_0(y)}{f_0(x)} \leq \left(\frac{y}{x}\right)^{\frac{a_{n+1}}{a_n}} \quad \text{for every } n \in \mathbb{N}. \quad (1.3)$$

In particular, $a_{n-1} < a_{n+1}$ for every $n \in \mathbb{N}$. Finally, using Lemma 3.3 of [4] we get $\alpha > 2$ and passing with n to $+\infty$ in (1.3) we obtain (1.2).

(ii) Define a function $g_0: [\min\{0, \log c\}, \max\{0, \log c\}] \rightarrow \mathbb{R}$ by setting $g_0(x) = \log f_0(e^x)$. Then g_0 is continuous, $g_0(0) = \log c$, $g_0(\log c) = \alpha \log c$ and $\frac{1}{\beta} \leq \frac{g_0(y) - g_0(x)}{y - x} \leq \beta$ for all $x, y \in [\min\{0, \log c\}, \max\{0, \log c\}]$ with $x < y$.

Since $0 < \frac{1}{\beta} < 1 < \beta$, we conclude from Theorem 2 of [5] that g_0 can be uniquely extended to a continuous function $g: \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$g^2(x) = \alpha g(x) - x \quad \text{for every } x \in \mathbb{R}, \quad (1.4)$$

and this unique extension g satisfies

$$\begin{aligned} g(x) &< \min \left\{ \beta x, \frac{1}{\beta} x \right\} && \text{for every } x \in \mathbb{R}, \text{ if } c \in (0, 1), \\ g(x) &> \max \left\{ \beta x, \frac{1}{\beta} x \right\} && \text{for every } x \in \mathbb{R}, \text{ if } c \in (1, +\infty), \\ \frac{1}{\beta}(y - x) &\leq g(y) - g(x) \leq \beta(y - x) && \text{for all } x, y \in \mathbb{R} \text{ with } x < y. \end{aligned}$$

Consequently, g is strictly increasing and maps \mathbb{R} onto \mathbb{R} . This jointly with (1.4) implies $g(x) + g^{-1}(x) = \alpha x$ for every $x \in \mathbb{R}$.

Now define a function $f: (0, +\infty) \rightarrow (0, +\infty)$ by setting $f(x) = e^{g(\log x)}$. Then f is an increasing bijection satisfying (1.1) such that (a) and (b) hold. The proof of the theorem is complete. \square

Remark 2. An answer to the problem posed during The Forty-ninth International Symposium on Functional Equations by Nicole Brillouët-Belluot (see [2]) as well as the result presented during The Fiftieth International Symposium on Functional Equations by Zoltán Boros (see [1]) can be derived from Theorem 10 of [5] in the same way as above (involving Lemmas 2.1–2.5 of [3] in the case of the problem of Nicole Brillouët-Belluot).

Acknowledgments

I thank Professor Karol Baron for calling my attention to the paper by Seiji Nabeya. This research was supported by Silesian University Mathematics Department (Iterative Functional Equations and Real Analysis program).

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Janusz Morawiec
Institute of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice
Poland
e-mail: morawiec@math.us.edu.pl

Received: October 22, 2013

Revised: December 12, 2013